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# Decomposition of Hilbert space in sets of coherent states

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**Abstract**

Within the generalized definition of coherent states as group orbits we study the orbit spaces and the orbit manifolds in the projective spaces constructed from linear representations. Invariant functions are suggested for arbitrary groups. The group $SU(2)$ is studied in particular and the orbit spaces of its $j = 1/2$ and 1 representations completely determined. The orbits of $SU(2)$ in $CP^N$ can be either two- or three-dimensional, the former being either isomorphic to $S^2$ or to $RP^2$ and the latter being isomorphic to quotient spaces of $RP^3$. We end with a look from the same perspective to the quantum mechanical space of states in particle mechanics.

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## 1. Introduction

Coherent states are an important tool in the study of wave phenomena finding many relevant applications in quantum physics [1, 2], both in particle mechanics and in field theory [3–6]. The familiar Glauber states [7, 8] can be equivalently defined as the elements of the orbit of the Heisenberg–Weyl group which contains the ground state, as the eigenstates of the annihilation operator or as the minimum uncertainty wavepackets. Following these different definitions there are different approaches to the generalization of the concept of coherent states. Here we privilege the group theoretical approach [9]. The generalization procedure has been extended to include systems with no classical analogue such as spin systems [10, 11] and others [12–16]. For a fuller account of applications of coherent states in different areas of physics see [17], where a more complete and historical list of references can be found.

In the group theoretical approach to coherent states Hilbert space is decomposed into the union of disjoint sets of coherent states, the group orbits. In finite-dimensional Hilbert spaces the orbits can be labelled using invariant (in the sense that they are constant within orbits) real functions in Hilbert space. These functions together with the group parameters completely parametrize Hilbert space. The-dimensionality of the sets of coherent states can be related to the values these invariant functions have on the sets.

Here we apply known results from group theory and invariant theory (reviewed in section 2 together with appendix A) to the study of coherent states as group orbits (reviewed in section 3) in the complex projective spaces of quantum mechanics (see appendix B). We make a proposal for invariant polynomial functions constructed from the Casimir operators in section 4.

The group $SU(2)$ is studied in detail in section 5. Orbits turn out to be either two- or three-dimensional; the former are in a finite number ($\mathrm{int}(j+1)$) within each irreducible representation $j$ and they are either isomorphic to $S^2$ or to $RP^2$ (section 5.1). In section 5.2 we work out completely the $j = 1$ representation: the orbit space is isomorphic to a line segment; the orbits in its interior are isomorphic to the three-dimensional lens space $S^3/Z^4$ and on its vertices they are two-dimensional (one isomorphic to $S^2$ and the other to $RP^2$); the invariant function $\overline{J_i}\,\overline{J_i}$ serves as a label for the orbits. Our results confirm those of [18] where they overlap. We comment on possible approaches to the study of higher $j$ representations using analytical as well as numerical methods in section 5.3. We end this section comparing our results for the two-dimensional orbits with the known formulae for coherent states in $SU(2)$ systems (section 5.4).

We finish in section 6 with the definition of invariants for the infinite-dimensional Hilbert spaces of particle mechanics.

## 2. Group orbits and invariants

Here we review the mathematical background about group orbits and how to label them using real functions which are invariants on the orbits. This subject can be found in the mathematical literature for group theory and invariant theory [19–21] and it has been explored in physics mostly in the study of the minima of potential functions in theories with spontaneous symmetry breaking where these potentials are invariant functions in the representation space of the gauge group [22–26].

Let $U(g)$ be a representation of the Lie group $G$ with Lie algebra $\mathcal{G}$ on the manifold $\mathcal{H}$. We represent points in $\mathcal{H}$ by $|\psi\rangle$, anticipating the application to vector spaces that we have in mind. The $G$-orbit through $|\phi\rangle$ is the subset of $\mathcal{H}$ given by

$$\mathcal{C}_\phi = \{|\psi\rangle \in \mathcal{H} : |\psi\rangle = U(g)|\phi\rangle,\ g \in G\}\,. \tag{1}$$

If the group $G$ is smooth and compact, the $G$-orbits are smooth, closed and compact submanifolds of $\mathcal{H}$. They are also connected if $G$ is connected. The relation '$|\phi'\rangle$ lies on the same orbit as $|\phi\rangle$' is clearly an equivalence relation: reflexive, symmetric and transitive. As a consequence $\mathcal{H}$ can be partitioned into disjoint orbits

$$\mathcal{H} = \bigcup_\phi \mathcal{C}_\phi \tag{2}$$

where the label $\phi$ runs over orbits (equivalence classes) and not over points. The quotient space $\mathcal{H}/G$ is called the orbit space.

The little group (or isotropy group) of $G$ at $|\phi\rangle$ is the subgroup $G_\phi$ of $G$ which leaves $|\phi\rangle$ fixed:

$$G_\phi = \{g \in G : U(g)|\phi\rangle = |\phi\rangle\}\,. \tag{3}$$

The subgroup $G_\phi$ is a Lie group and it may not be connected even if $G$ is. Its Lie algebra is formed by the elements of $\mathcal{G}$ which annihilate $|\phi\rangle$:

$$\mathcal{G}_\phi = \{t \in \mathcal{G} : t|\phi\rangle = 0\}\,. \tag{4}$$

The little groups at points lying on the same orbit are conjugated in $G$ and are therefore isomorphic; if $|\phi'\rangle = U(g)|\phi\rangle$ then

$$G_{\phi'} = gG_\phi g^{-1}. \tag{5}$$

Consequently the dimension of each orbit is

$$\dim \mathcal{C}_\phi = \dim G - \dim G_\phi. \tag{6}$$

The class of all subgroups of $G$ conjugated in $G$ to $G_\phi$ forms an equivalence class, the orbit type $\Omega_\phi$. Distinct-orbit types are disjoint. In the set of all orbit types a partial-ordering relation can be introduced: $\Omega_{\phi'} \leqslant \Omega_\phi$ if an element of $\Omega_{\phi'}$ is conjugated to a proper subgroup of an element of $\Omega_\phi$, and we say that $\Omega_{\phi'}$ has a lower symmetry than $\Omega_\phi$. An orbit is said to be principal if $\Omega$ is locally minimal in orbit space. A point is said to be principal if it lies on a principal orbit. The set of all orbits with the same orbit type $\Omega$ is called a stratum.

A function $f(|\psi\rangle)$ in representation space $\mathcal{H}$ is said to be $G$-invariant if

$$f(U(g)|\psi\rangle) = f(|\psi\rangle) \qquad \forall g \in G \qquad \forall |\psi\rangle \in \mathcal{H}. \tag{7}$$

It follows that $G$-invariant functions are also functions on the orbit space $\mathcal{H}/G$.

In appendix A we show some results and techniques applicable to real orthogonal linear representations (not necessarily irreducible) of compact groups. We are interested, for quantum mechanical applications, in complex unitary linear representations. But there is a standard correspondence between any unitary $n$-dimensional complex representation $U(g)$ of $G$ and an orthogonal $2n$-dimensional real representation $\mathrm{O}(g)$, called the realification of $U(g)$, which is obtained by the simple correspondence between vectors

$$(z_1, z_2, \ldots, z_n) \leftrightarrow (\operatorname{Re} z_1, \operatorname{Re} z_2, \ldots, \operatorname{Re} z_n, \operatorname{Im} z_1, \operatorname{Im} z_2, \ldots, \operatorname{Im} z_n). \tag{8}$$

Since $U(N) = U(1) \times SU(N)$ all vectors in a Hilbert space carrying a non-trivial (in the $U(1)$ factor) representation of $U(N)$ which differ solely by a phase factor lie on the same orbit. Therefore the orbit space for the complex projective representations of $U(N)$ and $SU(N)$ are the same. For the same reason the orbit space of the complex projective representation of $U(N)$ is the same as the orbit space of the real projective representation of the realification of $U(N)$. Thus the orbit space of the complex projective representation $R$ of $SU(N)$ coincides with the projective slice of the realification of the representation $R \times S$ of $U(N)$, where $S$ is a non-trivial representation of $U(1)$. The orbits themselves have the same little groups and as manifolds they are copies of the orbits of $SU(N)$ in projective space multiplied by $S^1$ on account of all the vectors differing by a phase which are not identified in the latter representation.

We finish this section with a remark about the complex projective spaces $P\mathcal{H}$ obtained after the identifications (B.1) (see appendix B). Unitary transformations do not change the norm of a vector but they may change its phase. As a consequence, when using vectors $|\phi\rangle$ in complex vector spaces $\mathcal{H}$ to describe points in $P\mathcal{H}$, the Lie algebra of the little group $G_\phi$ is no longer given by the elements of $\mathcal{G}$ which annihilate $|\phi\rangle$ (4) but rather by its elements for which $|\phi\rangle$ is an eigenvector with real eigenvalue

$$\mathcal{G}_\phi = \{t \in \mathcal{G} : t|\phi\rangle = T|\phi\rangle, T \in R\}. \tag{9}$$

## 3. Coherent states as group orbits

We follow [9] and define a subset $\mathcal{C}$ of Hilbert space $\mathcal{H}$ to be a set of coherent states if it is continuous (and we represent its elements by $|c\rangle$, $c$ denoting a finite number of continuous parameters) and if there exists a positive measure d$c$ on it admitting the partition of the unit operator

$$\int_{\mathcal{C}} \mathrm{d}|c\rangle\langle c| = 1. \tag{10}$$

Continuity guarantees that it is always possible to redefine the measure d$c$ in such a way that the $|c\rangle$ states are normalized.

For a one-particle system in mechanics the Glauber states [7, 8] can be written as

$$|q, p\rangle = U(q, p)|0\rangle \tag{11}$$

where $U(p, q)$ is the Weyl operator

$$U(q, p) = \mathrm{e}^{\mathrm{i}(pQ - qP)/\hbar}. \tag{12}$$

It can be shown that these are minimum-uncertainty states ($\Delta Q \Delta P = \hbar/2$) and that they are eigenstates of the annihilation operator. Their eigenvalues provide an useful analytic representation in the complex plane which allows moreover for a differential representation of operators acting on the functions $\langle c|\psi\rangle$. Further information on these properties can be found in the literature [17, 27]. Here we restrict attention to the group theoretical properties associated with coherent states.

The Weyl operators act as translation operators for position and momentum in the sense that

$$U^{+}(q, p) Q U(q, p) = Q + q \tag{13}$$

$$U^{+}(q, p) P U(q, p) = P + p. \tag{14}$$

It follows that

$$\langle q, p|Q|q, p\rangle = q \qquad \text{and} \qquad \langle q, p|P|q, p\rangle = p. \tag{15}$$

One can derive the properties

$$U(0, 0) = 1 \tag{16}$$

$$U^{-1}(q, p) = U^{+}(q, p) = U(-q, -p) \tag{17}$$

$$U(q_2, p_2) U(q_1, p_1) = \mathrm{e}^{\mathrm{i}(q_1 p_2 - p_1 q_2)/2\hbar} U(q_2 + q_1, p_2 + p_1) \tag{18}$$

which show that the Weyl operators form a group when acting on projective Hilbert space $P\mathcal{H}$ (see appendix B). On the whole of Hilbert space the Weyl operators together with an Abelian factor $\mathrm{e}^{\mathrm{i}\theta}$ form a group, the Heisenberg–Weyl group; that is, the Heisenberg–Weyl group is an extension of the set of the Weyl operators by a circle.

Sets of generalized coherent states in particle mechanics other than the Glauber states which fit the definition given at the beginning of this section can be constructed by applying the Weyl operators to an arbitrary vector $|\phi\rangle$ in Hilbert space $\mathcal{H}$ [17]

$$\mathcal{C}_\phi = \{|p, q; \phi\rangle = U(q, p)|\phi\rangle,\ (q, p) \in R^2\}. \tag{19}$$

Like the set of Glauber states, these sets admit a differential representation of operators. But they lack the analytic representation in the complex plane and they are not states of minimum uncertainty since the vector $|\phi\rangle$ that one starts from is arbitrary and it can have any values of variances $\Delta Q^2$ and $\Delta P^2$ *a priori*. They are not eigenstates of any particularly simple operator either.

This way of generating sets of coherent states as orbits of groups in Hilbert space has been generalized to representations of arbitrary Lie groups $G$ [14]. Let $U(g)$, $g \in G$, be an irreducible unitary representation of $G$ acting on the space $\mathcal{H}$. Pick any vector $|\phi\rangle \in \mathcal{H}$ and consider the $G$-orbit $\mathcal{C}_\phi$ (1) passing through $|\phi\rangle$. One can label the vectors in $\mathcal{C}_\phi$ with the group elements

$$\mathcal{C}_\phi = \{|g; \phi\rangle = U(g)|\phi\rangle,\ x \in G\}. \tag{20}$$

Continuity of the representation $U(g)$ ensures continuity of the set $\{|g; \phi\rangle\}$. In particular one has for the inner product

$$\langle g; \phi|g'; \phi\rangle = \langle\phi|U^{+}(g)U(g')|\phi\rangle = \langle\phi|U(g^{-1}g')|\phi\rangle \tag{21}$$

which is bounded by unity. Let d$g$ denote the left invariant measure on the group $G$. Then if

$$d = \int \mathrm{d}g |\langle\phi|U(g)|\phi\rangle|^2 \tag{22}$$

converges one has [14]

$$\frac{1}{d}\int \mathrm{d}g|g;\phi\rangle\langle g;\phi| = 1. \tag{23}$$

Therefore the sets $\mathcal{C}_\phi$ satisfy the criteria given at the beginning of this section to qualify as coherent states. Representations obeying (22) are termed square integrable and they are always so if the volume of group space $\int \mathrm{d}g$ is finite, which happens for compact groups. We emphasize that without further specification these sets of generalized coherent do not lead necessarily to analytic function representations [28].

From the definition of the orbits we see that the vectors $U(g)|\phi\rangle$ for all $g$ which belongs to one left coset of the little group $G_\phi$ in $G$ differ from one another at most by a phase factor and that these vectors determine the same state in complex projective space. Thus we may label the vectors in the orbit $\mathcal{C}_\phi$ with the elements $x$ of the coset space $X_\phi = G/G_\phi$ and we write

$$\mathcal{C}_\phi = \left\{|x;\phi\rangle = U[g(x)]|\phi\rangle,\ x \in X_\phi\right\} \tag{24}$$

where $g(x)$ is any representative $x$ of the coset. In this way we avoid including 'repeated' vectors in the representation of the orbit as it may be the case using the set $\{|g;\phi\rangle\}$. In many cases the measure d$g$ on $G$ induces an invariant measure d$x$ on $X_\phi = G/G_\phi$, which may not be unique. Then the inner product (21) and the partition of identity (23) become

$$\langle x;\phi|x';\phi\rangle = \langle\phi|U[g(x)^{-1}g(x')]|\phi\rangle \tag{25}$$

$$1 = \frac{1}{d'}\int \mathrm{d}x|x;\phi\rangle\langle x;\phi| \tag{26}$$

where

$$d' = \int \mathrm{d}x|\langle\phi|U[g(x)]|\phi\rangle|^2. \tag{27}$$

Both (21)–(23) and (25), (26) are correct and it is somewhat a matter of taste which one is preferred (provided that d$x$ is well defined). We shall use mostly the second form.

Let us now specialize to the group $SU(2)$ which admits representations classified according to integer and semi-integer values $j$ with the Casimir operator $J^2 = j(j+1)\hbar^2$. Let $\mathcal{H}$ be a Hilbert space carrying one such representation. The sets of coherent states (20) are obtained by acting on any fiducial state $|\phi\rangle \in \mathcal{H}$ with the group elements of $SU(2)$. Using the group parametrization

$$U(z,\theta) = N\mathrm{e}^{zJ_-/\hbar}\mathrm{e}^{-z^*J_+/\hbar}\mathrm{e}^{-\mathrm{i}\theta J_z/\hbar} \tag{28}$$

where $J_\pm$ are the ladder operators $J_\pm = J_x \pm \mathrm{i}J_y$, and choosing the fiducial state to be an eigenstate of $J_z$, $|m\rangle$ with $m = -j,\ldots,j$, one gets the following set of coherent states [11]:

$$|z;m\rangle = U(z)|z\rangle = N\mathrm{e}^{zJ_-/\hbar}\mathrm{e}^{-z^*J_+/\hbar}|m\rangle \tag{29}$$

where the phase factor resulting from $\mathrm{e}^{-\mathrm{i}\theta J_z/\hbar}$ has been ignored (this corresponds to using (24) rather than (20)) and $N$ stands for a normalization factor. Further choosing $|j\rangle$ as the fiducial state one has $\mathrm{e}^{-z^*J_+/\hbar}|j\rangle = |j\rangle$ and

$$|z\rangle = \frac{1}{(1+|z|^2)^j}\mathrm{e}^{zJ_-}|j\rangle \tag{30}$$

after determination of the normalization factor. This analytic representation is not available in general for the sets (20) generated from arbitrary fiducial vectors. Uncertainty is not constant within sets of spin coherent states according to [29] but it may be so according to other definitions (see [30] where the sets (30) turn out to be composed of minimum-uncertainty states).

## 4. Invariants for projective representations

In order to construct real functions which are invariant within orbits (24) we make use of the inner product in Hilbert space. Clearly the inner product itself $\langle x; \phi | x; \phi \rangle$ is such an invariant. It can be used to label orbits on the whole of Hilbert space but we are restricting attention to projective space where $\langle x; \phi | x; \phi \rangle = 1$ is a constant. Consider the generalized Casimir operators [31]

$$C_n = c^{b_2}_{a_1 b_1} c^{b_3}_{a_2 b_2} \ldots c^{b_1}_{a_n b_n} X^{a_1} X^{a_2} \ldots X^{a_n} \tag{31}$$

where $c^c_{ab}$ are the structure constants of the Lie algebra $\mathcal{G}$ and $X_a$ its generators

$$[X_a, X_b] = c^c_{ab} X_c. \tag{32}$$

Indices are raised and lowered in the Lie algebra using the metric $g_{ab} = c^d_{ac} c^c_{bd}$. The generators of the algebra transform under the action of the group according to the adjoint representation $A^a_b(g)$

$$U^+(g) X^a U(g) = A^a_b(g) X^b. \tag{33}$$

Since the Casimir operators commute with all generators of the algebra one has

$$U^+(g) C_n U(g) = A^{a_1}_{c_1}(g) c^{b_2}_{a_1 b_1} \ldots A^{a_n}_{c_n}(g) c^{b_1}_{a_n b_n} X^{c_1} \ldots X^{c_n} = C_n. \tag{34}$$

As a consequence the mean value of any Casimir operator $\langle x; \phi | C_n | x; \phi \rangle = \overline{C_n}(x; \phi)$ is an invariant within orbits. But it is of no use to parametrize the orbits because it is actually constant within the whole irreducible representation. Notice, however, that for any polynomial in the generators of the algebra one has

$$\begin{aligned} \overline{X^{a_1} \ldots X^{a_p}}(x; \phi) &= \langle \phi | U^+[g(x)] X^{a_1} \ldots X^{a_p} U[g(x)] | \phi \rangle \\ &= A^{a_1}_{b_1}[g(x)] \ldots A^{a_n}_{b_n}[g(x)] \overline{X^{b_1} \ldots X^{b_p}}(\phi). \end{aligned} \tag{35}$$

Then according to (34) any function of the form

$$f = c^{b_2}_{a_1 b_1} c^{b_3}_{a_2 b_2} \ldots c^{b_1}_{a_n b_n} \overline{X^{a_1} X^{a_2}}\; \overline{X^{a_3}}\; \overline{X^{a_4} X^{a_5} X^{a_6}} \ldots \overline{X^{a_n}} \tag{36}$$

where the mean values are evaluated over any combinations of the generators $X^a$ is an invariant within orbits. It is clear that using the commutator (32) one can express any function of this form as a linear combination of functions of the same type which are real. To make clear what do we mean with (36) let us give the example of the quartic Casimir operator from which the following invariant functions can be constructed:

$$f_1 = c^{b_2}_{a_1 b_1} c^{b_3}_{a_2 b_2} c^{b_4}_{a_3 b_3} c^{b_1}_{a_4 b_4} \overline{X^{a_1} X^{a_2} X^{a_3} X^{a_4}} \tag{37}$$

$$f_2 = c^{b_2}_{a_1 b_1} c^{b_3}_{a_2 b_2} c^{b_4}_{a_3 b_3} c^{b_1}_{a_4 b_4} \overline{X^{a_1} X^{a_2} X^{a_3}}\; \overline{X^{a_4}} \tag{38}$$

$$f_3 = c^{b_2}_{a_1 b_1} c^{b_3}_{a_2 b_2} c^{b_4}_{a_3 b_3} c^{b_1}_{a_4 b_4} \overline{X^{a_1} X^{a_2}}\; \overline{X^{a_3} X^{a_4}} \tag{39}$$

$$f_4 = c^{b_2}_{a_1 b_1} c^{b_3}_{a_2 b_2} c^{b_4}_{a_3 b_3} c^{b_1}_{a_4 b_4} \overline{X^{a_1} X^{a_2}}\; \overline{X^{a_3}}\; \overline{X^{a_4}} \tag{40}$$

$$f_5 = c^{b_2}_{a_1 b_1} c^{b_3}_{a_2 b_2} c^{b_4}_{a_3 b_3} c^{b_1}_{a_4 b_4} \overline{X^{a_1}}\; \overline{X^{a_2}}\; \overline{X^{a_3}}\; \overline{X^{a_4}}. \tag{41}$$

The first of these functions is the mean value of the quartic Casimir operator which we know to be a constant throughout all of Hilbert space, but there is no reason *a priori* why the remaining functions should have the same value at different orbits. On the other hand it is obvious that the functions $f$ of the generic form (36) cannot all be independent in orbit space. At most $N$ of them can be so, $N$ being the dimension of orbit space. Our conjecture is that there can indeed be found $N$ such functions which separate the orbits in projective space and the values of these functions can then be used to parametrize the orbits.

## 5. The group $SU(2)$

### *5.1. General setting and the two-dimensional representation*

Here we propose to study the orbit space and the invariants for the complex projective representations of $SU(2)$. A similar task has been carried out for the linear representations of $SU(2)$ in [32] and of $SO(3)$ in [33]. Our problem is related to these but different, and it has been studied in [18]. Our presentation is complementary to [18] both in the methods used and in the results. For the projective representations of the group $SU(2)$ the element $g = -1$, that is the rotation by $2\pi$, always belongs to the little group of any vector. Therefore these representations can also be seen as representations of $SO(3)$. We shall for simplicity omit the factor $\{1, -1\}$ in the little groups or, which is the same, look upon the spaces as representations of $SO(3)$. In this section we take $\hbar = 1$ for simplicity. We shall also consider in the remaining of this section that $j \neq 0$; the analysis of the identity representation is trivial and in many respects singular.

Let $J_i$ $(i = 1, 2, 3)$ be the generators of the Lie algebra of $SO(3)$ with commutation relations

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k. \tag{42}$$

The quadratic Casimir operator is

$$J^2 = J_i J_i. \tag{43}$$

The higher-order Casimir operators in (31) are powers of $J^2$ and consequently we can think of the invariants of the type (36) as constructed from powers of $J^2$. It is easy to see that up to the third power in $J^2$ all the invariants of the type (36) can be written in terms of the following eight:

$$\begin{aligned}
&f_1 = \overline{J_i}\;\overline{J_j} && f_2 = \overline{J_i}\;\overline{J_j}\;\overline{J_i J_j} && f_3 = \overline{J_i J_j}\;\overline{J_j J_i} \\
&f_4 = \overline{J_i}\;\overline{J_j}\;\overline{J_i J_k}\;\overline{J_k J_j} && f_5 = \overline{J_i J_j}\;\overline{J_j J_k}\;\overline{J_k J_i} && f_6 = \overline{J_i}\;\overline{J_j}\;\overline{J_k}\;\overline{J_i J_j J_k} \\
&f_7 = \overline{J_i}\;\overline{J_j J_k}\;\overline{J_j J_i J_k} && f_8 = \overline{J_i J_j J_k}\;\overline{J_k J_j J_i}.
\end{aligned} \tag{44}$$

All other orderings of operators can be written in terms of these using the commutator (42). These functions are real and they will be enough for the applications of the remaining sections.

The Lie algebra of the little group is given by the elements satisfying (9)

$$\vec{r} \cdot \vec{J} |\psi\rangle = \lambda |\psi\rangle. \tag{45}$$

In other words, if $|\phi\rangle$ is not an eigenvector of angular momentum in some direction, then the Lie algebra of $G_\phi$ is trivial $\mathcal{G}_\phi = \{0\}$ and the dimension of the orbit $\mathcal{C}_\phi$ is maximal, that is $\dim \mathcal{C}_\phi = 3$ because the group $SO(3)$ is three dimensional. On the other hand, if $|\phi\rangle$ is an eigenvector of angular momentum in some direction, it cannot be so in any other direction and the Lie algebra of its little group is generated by the operator of angular momentum $\hat{r}_\phi \cdot \vec{J}$ in that particular direction $\hat{r}_\phi$ for which $|\phi\rangle$ is an eigenvector. Therefore the connected component of the little group $G_\phi$ is the subgroup of rotations around the axis in the direction $\hat{r}_\phi$. This is a one-dimensional subgroup and consequently the orbits are two dimensional. We conclude that for $SO(3)$ there are only two- and three-dimensional orbits. The first consist of all vectors which

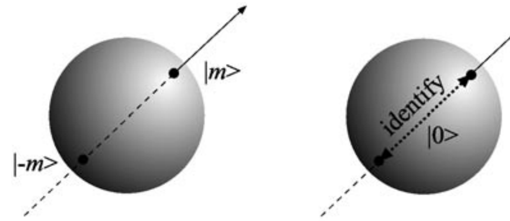


**Figure 1.** The two-dimensional orbits include one and only one of the vectors $|m\rangle$ with $m \geqslant 0$. There are $2j$ orbits isomorphic to $S^2$ (left) for $m \neq 0$, and if $j$ is an integer one orbit isomorphic to $RP^2$ (right) for $m = 0$.

are eigenvectors of angular momentum $\hat{r} \cdot \vec{J}$ in some direction $\hat{r}$. Note that these considerations apply only to the connected part of the little group; there may be non-trivial discrete factors multiplying the connected part of the little group. In fact as we shall see the little group is in general not connected and orbits with the same dimensionality may differ in their little groups and therefore not be isomorphic.

Consider a three-dimensional orbit. If its little group is trivial then each element of $SO(3)$ defines one point in the orbit, and $\mathcal{C}_\phi$ is isomorphic to $SO(3)$ which is in turn isomorphic to three-dimensional real projective space $RP^3$. If the little group is not trivial, $\mathcal{C}_\phi$ is isomorphic to the coset space $SO(3)/G_\phi$ which is to say to a quotient space of $RP^3$ by a discrete group.

The two-dimensional orbits can be worked out in detail in the general case. We know that the eigenvalues of angular momentum in the $z$-direction $J_z$ are finite and non-degenerate:

$$J_z|m\rangle = m|m\rangle \qquad \text{with} \quad m = -j, -j+1, \ldots, j-1, j \tag{46}$$

and that

$$\langle m|\vec{J}|m\rangle = m\vec{e}_z. \tag{47}$$

Applying an element of $SO(3)$ to $|m\rangle$ clearly brings it to the eigenvector of the rotated direction with eigenvalue $m$. Since these eigenvectors are not degenerate this means that all states belonging to a two-dimensional orbit can be generated after a rotation from one of the vectors $|m\rangle$, or which is the same that all orbits contain at least one of the vectors $|m\rangle$. It is clear also from the non-degeneracy of the eigenvectors that after a rotation by $\pi$ around any axis orthogonal to the $z$-axis the vector $|m\rangle$ is mapped to $|-m\rangle$. As a consequence for $m = 0$ these rotations also belong to the little group of $|0\rangle$. On the other hand they do not for $m \neq 0$ but one realizes that $|m\rangle$ and $|-m\rangle$ belong to the same orbit. Moreover eigenvectors in different (non-parallel) directions cannot be identical. We conclude that there is a finite number of two-dimensional orbits which can be generated from the vectors $|m\rangle$ with $m \geqslant 0$. For $m > 0$ the little group is the subgroup of rotations around the quantization axis $G_m = R_z$ and the orbit space consists of all possible directions, which is topologically the two-sphere $S^2$. For $m = 0$ the little group is $R_z$ plus the rotations by $\pi$ in directions orthogonal to the quantization axis $R_{(x,y)}(\pi)$, $G_0 = R_z + R_{(x,y)}(\pi) = R_z \times R_x(\pi)$ and the orbit space consists of all possible directions (up to sign), which is topologically the two-dimensional projective space $RP^2$. The first invariant in (45) can be used to distinguish the different two-dimensional orbits since $f_1(|m\rangle) = m^2$. In figure 1 we depict the two types of two-dimensional orbits and in figures 2 and 3 we represent their respective little groups (known in the mathematical literature as $C_\infty$ and $D_\infty$).

The projective space associated to the representation $j$ is $CP^{2j}$:

$$j \;\rightarrow\; \dim \mathcal{H} = 2j+1 \;\rightarrow\; \text{projective space} : CP^{2j} \tag{48}$$

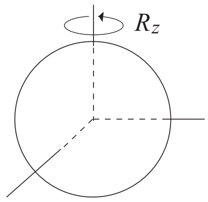


**Figure 2.** The little group for the $S^2$ orbits of $SU(2)$.

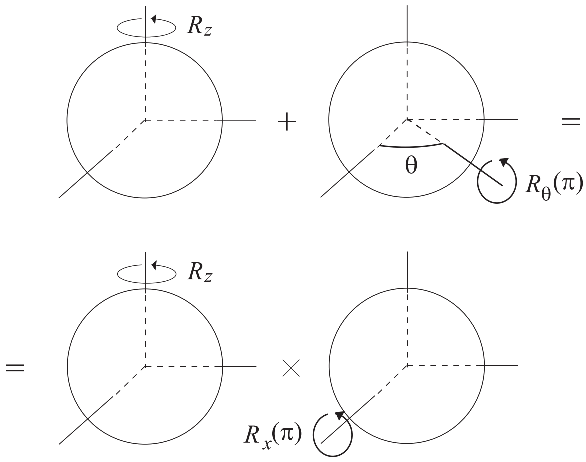


**Figure 3.** The little group for the $RP^2$ orbits of $SU(2)$.

and its real dimension is $4j$ (see appendix B). Its dimension is therefore greater than 2 for $j > 1/2$ and since the two-dimensional orbits are in a finite number, most of $CP^{2j}$ must consist of points belonging to three-dimensional orbits. Thus the dimension of orbit space is $4j - 3$. For $j = 1/2$ one has $\dim CP^1 = 2$ and there can be no three-dimensional orbits. On the other hand we know that there is only one two-dimensional orbit for $m = 1/2$. Therefore the whole of $CP^1$ consists of a single two-dimensional orbit isomorphic to $S^2$. This is in agreement with the known isomorphism between $CP^1$ and $S^2$.

We summarize this analysis of orbit space in the following three statements:

(I) The orbit space of $SU(2)$ is $(4j - 3)$-dimensional for its irreducible representations with $j > 1/2$ and consists of three-dimensional orbits apart from a finite number of elements which are two-dimensional orbits. The orbit space of the representation $j = 1/2$ consists of one single point.

(II) The three-dimensional orbits are topologically isomorphic to quotient spaces of $RP^3$.

(III) The two-dimensional orbits are in number of $\underline{\mathrm{int}}(j+1)$ (integer part of $j+1$) and they can be distinguished by the value of the invariant $\overline{J_i}\,\overline{J_i} = j^2, (j-1)^2, \ldots$ with minimum value $1/4$ for semi-integer $j$ representations and 0 for integer $j$ representations. Topologically these orbits are isomorphic to two-spheres $S^2$ except for the $\overline{J_i}\,\overline{J_i} = 0$ orbit of integer $j$ representations which is isomorphic to the two-dimensional real projective space $RP^2$.

The possible little groups of the elements of the three-dimensional orbits can be found in [18].

*5.2. The $j = 1$ representation*

The projective space of the representation $j = 1$ is four-dimensional $CP^2$. Using the results of the previous section we can state that the orbit space is one-dimensional and consists of three-dimensional orbits plus two two-dimensional orbits, one isomorphic to $S^2$ ($m = 1$) and the other to $RP^2$ ($m = 0$). In terms of a $G$-invariant function $f : CP^2 \to R$ labelling the orbits, these two two-dimensional orbits must lie at the vertices of the image of $f$ in $R$. Therefore one can state that orbit space is a line segment. Its interior must be of one orbit type only (the principal stratum, see appendix A) for which the orbits are some quotient space of $RP^3$. As a first guess for the $G$-invariant function to label orbits we may take the function $f_1 = \overline{J_i}\,\overline{J_i}$ in (45).

Now we proceed to the explicit computation of orbits using canonical group coordinates

$$U(\vec{r}) = \mathrm{e}^{\mathrm{i}\vec{r}\cdot\vec{\sigma}} \tag{49}$$

with

$$\sigma_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \sigma_y = \frac{\mathrm{i}}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{50}$$

and the representation of $CP^2$ given by vectors of the form

$$\begin{bmatrix} \sin\theta_1 \sin\theta_2 \mathrm{e}^{\mathrm{i}\beta_1} \\ \cos\theta_1 \\ \sin\theta_1 \cos\theta_2 \mathrm{e}^{\mathrm{i}\beta_2} \end{bmatrix} \tag{51}$$

for which

$$f_1 = \sin^2\theta_1\left[\sin^2\theta_1(\cos^2\theta_2 - \sin^2\theta_2)^2 + 8\cos^2\theta_1\cos\theta_2\sin\theta_2\cos(\beta_1+\beta_2)\right]. \tag{52}$$

One has $0 \leqslant f_1 \leqslant 1$.

The eigenvalue equation (45) has two families of solutions

$$|\alpha, \beta; 1\rangle = \begin{bmatrix} \cos^2\alpha \mathrm{e}^{-\mathrm{i}\beta} \\ \sin(2\alpha)/\sqrt{2} \\ \sin^2\alpha \mathrm{e}^{\mathrm{i}\beta} \end{bmatrix} \tag{53}$$

$$|\alpha, \beta; 0\rangle = \begin{bmatrix} -\sin(2\alpha)\mathrm{e}^{-\mathrm{i}\beta}/\sqrt{2} \\ \cos(2\alpha) \\ \sin(2\alpha)\mathrm{e}^{\mathrm{i}\beta}/\sqrt{2} \end{bmatrix} \tag{54}$$

with ranges $\alpha \in (0, \pi/2)$ and $\beta \in (0, 2\pi)$. They represent the eigenvalues corresponding to the direction

$$\hat{r} = (\sin 2\alpha\cos\beta, \sin 2\alpha\sin\beta, \cos 2\alpha). \tag{55}$$

The parameter $\beta$ degenerates completely both at $\alpha = 0$ and at $\alpha = \pi/2$. In (54) states related by $\alpha \to \pi/2 - \alpha,\ \beta \to \beta + \pi$ correspond to the same point in $CP^2$. The first solution (53) is the expected $S^2$ orbit and the second one (54) is the $RP^2$ orbit. The vectors lying at $\alpha = 0$ and $\pi/2$ are easily recognizable as the eigenvectors $|1\rangle$ and $|-1\rangle$, respectively, in (53) and to correspond both to the eigenvector $|0\rangle$ in (54).

Now we check whether $f_1$ separates the orbits. We notice that any state belongs to the orbit of some state for which

$$\langle\psi|\vec{J}|\psi\rangle = J_z\vec{e}_z \qquad \text{with} \quad J_z \geqslant 0 \tag{56}$$

since it is always possible to rotate a vector and bring it to point in the positive $z$-direction. Therefore the solution to (56) contains at least one representative of each orbit. The solution

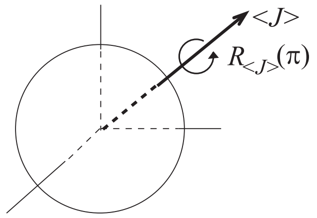


**Figure 4.** The little group for the three-dimensional orbits $S^3/Z^4$ of the three-dimensional representation of $SU(2)$.

to this equation consists of (54) which we know to be composed of one single orbit plus the set

$$|\theta,\beta\rangle = \begin{bmatrix} \cos\theta\,\mathrm{e}^{\mathrm{i}\beta} \\ 0 \\ \sin\theta\,\mathrm{e}^{-\mathrm{i}\beta} \end{bmatrix} \tag{57}$$

with $\theta \in [0, \pi/4]$, $\beta \in (0, 2\pi)$. But

$$|\theta,\beta\rangle = R_z(\beta)|\theta\rangle \qquad \text{with} \quad |\theta\rangle = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix}. \tag{58}$$

Moreover the vector $|\theta\rangle$ with $\theta = \pi/4$ belongs to the orbit (54). Consequently among the vectors $|\theta\rangle$ we still have at least one representative of each orbit. Now we compute

$$f_1(|\theta\rangle) = \cos^2\theta - \sin^2\theta = \cos(2\theta). \tag{59}$$

Clearly the map $f_1 : \theta \in [0, \pi/4] \mapsto [0, 1]$ is one-to-one. Thus it is demonstrated that $f_1$ separates the orbits. The two two-dimensional orbits (53) and (54) lie at the extrema of the line segment $f_1 \in [0, 1]$ as predicted

$$f_1(|\alpha,\beta;0\rangle) = 0 \qquad \text{and} \qquad f_1(|\alpha,\beta;1\rangle) = 1. \tag{60}$$

It remains to compute the little group of the orbits lying in the interior of $f_1 \in [0, 1]$. We can do this by direct calculation using the representatives $|\theta\rangle$ of (58) and the explicit form of (49) for $j = 1$ [35]

$$U(\vec{r}) = 1 + \frac{\mathrm{i}\sin r}{r}\begin{bmatrix} z & c^* & 0 \\ c & 0 & c^* \\ 0 & c & -z \end{bmatrix} + \frac{\cos r - 1}{r^2}\begin{bmatrix} z^2+|c|^2 & zc^* & c^{*2} \\ zc & 2|c|^2 & -zc^* \\ c^2 & -zc & z^2+|c|^2 \end{bmatrix} \tag{61}$$

where $r^2 = x^2 + y^2 + z^2$ and $c = (x + \mathrm{i}y)/\sqrt{2}$. The result is $G_\theta = \{1, R_z(\pi)\}$ for $\theta \in ]0, \pi/4[$, that is the discrete subgroup whose only non-trivial element is the rotation by $\pi$ around the $z$-axis. By symmetry it is clear that the little group for any other vector $|\psi\rangle$ belonging to a three-dimensional orbit is

$$G_\psi = \{1, R_{\langle\psi|\vec{J}|\psi\rangle}(\pi)\}. \tag{62}$$

This is depicted in figure 4. We confirm the expectation that the interior of the line segment $f_1 \in [0, 1]$ consists of one single stratum of three-dimensional orbits. Each orbit is a lens space with the topology of the quotient of the three-sphere by the cyclic group of order 4 [34]

$$\mathcal{C} = RP^3/Z^2 = S^3/Z^4. \tag{63}$$

We arrived at a picture of $CP^2$ as the product of a line segment by $S^3/Z^4$ manifolds which degenerate to $S^2$ at one extremum of the segment and to $RP^2$ at the other one (figure 5).

The remaining $G$-invariant functions in (45) are polynomials in $f_1$ as expected:

$$\begin{matrix} f_2 = f_1 & f_3 = 2 & f_4 = f_1 & f_5 = 2 & f_6 = {f_1}^2 \\ f_7 = f_1 & f_8 = 2 + f_1. & & & \end{matrix} \tag{64}$$

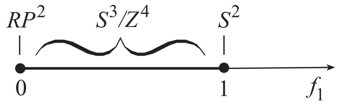


**Figure 5.** Orbit space for the three-dimensional representation of $SU(2)$.

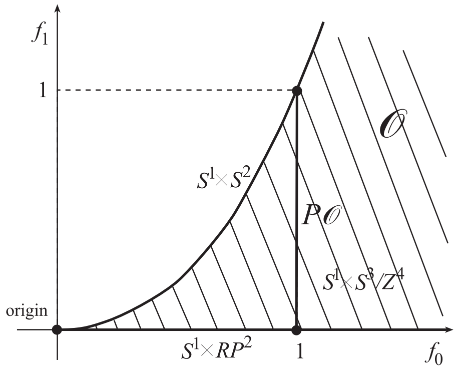


**Figure 6.** The orbit space for the representation $j = 1$ of $SU(2)$ as the projective slice $P\mathcal{O}$ of the orbit space $\mathcal{O}$ of the linear representation of the realification of $U(2)$.

### *5.3. The $j = 3/2$ representation and perspectives for future work*

In order to study the matrix $\hat{P}_{ij}$ of (A.3) we consider the whole Hilbert space of the representation of the realification of $U(2)$ and the two $G$-invariant functions $f_1$ and $f_0 = \langle\psi|\psi\rangle$ which separate the orbits. We have then

$$\hat{P} = \begin{bmatrix} \vec{\nabla} f_0 \cdot \vec{\nabla} f_0 & \vec{\nabla} f_0 \cdot \vec{\nabla} f_1 \\ \vec{\nabla} f_1 \cdot \vec{\nabla} f_0 & \vec{\nabla} f_1 \cdot \vec{\nabla} f_1 \end{bmatrix} = \begin{bmatrix} 4f_0 & 8f_1 \\ 8f_1 & 16f_2 \end{bmatrix} = \begin{bmatrix} 4f_0 & 8f_1 \\ 8f_1 & 16f_0 f_1 \end{bmatrix} \tag{65}$$

where the last equality is easily obtained from (64) generalizing this equations to $\mathcal{H}$ by-dimensional arguments. The values of $f_0$ and $f_1$ for which the matrix $\hat{P}$ is positive semi-definite satisfy

$$f_0 \geqslant 0 \qquad 0 \leqslant f_1 \leqslant {f_0}^2. \tag{66}$$

This is depicted in figure 6. There are four strata: the interior of this region is the principal stratum; the lines $\{f_0 > 0, f_1 = {f_0}^2\}$ and $\{f_0 > 0, f_1 = 0\}$ are two distinct strata composed respectively of $S^1 \times S^2$ and $S^1 \times RP^2$ orbits; and the point $\{f_0 = 0, f_1 = 0\}$ is the zero-dimensional stratum corresponding to the origin of Hilbert space. The slice $f_0 = \langle\psi|\psi\rangle = 1$ gives a faithful image of orbit space in the projective representation (see appendix A).

To use these techniques is one possible approach to study the higher-dimensional representations of $SU(2)$. We also performed some numerical calculations on the $j = 3/2$ representation. We leave these issues for possible future work. Here we exhibit in figures 7 and 8, as an example, the numerical plots for the projections of orbit space onto the planes $(f_1, f_2)$ and $(f_1, f_8)$ ($f_3 = f_1$ for the $j = 3/2$ representation). This representation contains only two two-dimensional orbits isomorphic to $S^2$ according to the results of section 5.1 lying at the points with values of $(f_1, f_2, f_8)$

$$\left(\tfrac{1}{4}, \tfrac{1}{16}, \tfrac{1}{64}\right) \qquad \text{and} \qquad \left(\tfrac{9}{4}, \tfrac{81}{16}, \tfrac{729}{64}\right). \tag{67}$$

In the figures one can observe the expected semi-algebraic variety nature of the image of orbit space. In particular one would expect the two-dimensional orbits to lie at vertices of the figures and indeed the kinks at the points (67) are visible in the graphics.

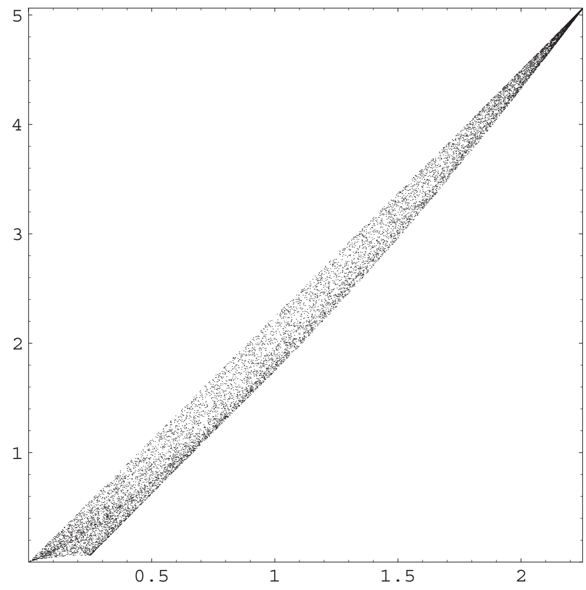


**Figure 7.** Numerical plot of the projection onto the plane $(f_1, f_2)$ of the image of orbit space for the $j = 3/2$ representation of $SU(2)$.

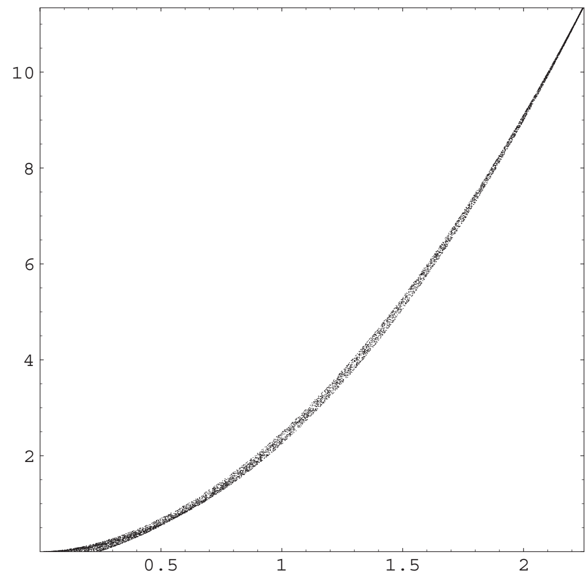


**Figure 8.** Numerical plot of the projection onto the plane $(f_1, f_8)$ of the image of orbit space for the $j = 3/2$ representation of $SU(2)$.

Numerics can also be used to study the shape of orbits in the picture of $CP^N$ described in appendix B. For the octant picture of $CP^2$, figure B.2, with $Z_0$ standing for the coordinate relative to the eigenvector $|0\rangle$ and $Z_1$ and $Z_2$ for the coordinates relative to the eigenvectors $|1\rangle$ and $|-1\rangle$, one realizes that the vertical projections of the orbits form rectangles with one side parallel to the bisectrix of the projected quadrant. The bisectrix itself is a degenerate rectangle

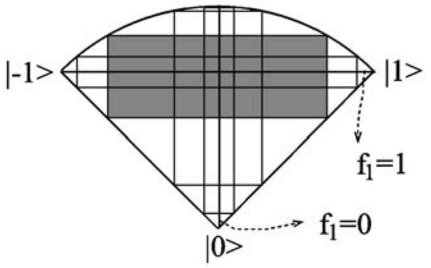


**Figure 9.** Orbits of the $j = 1$ representation of $SU(2)$ in the octant picture of $CP^2$ (vertical projection).

corresponding to the $RP^2$ orbit $f_1 = 0$. The other degenerate rectangle is the line joining the two opposed vertices of the quadrant and it corresponds to the $S^2$ orbit $f_1 = 1$. The function $f_1$ varies smoothly from one line to the other along the rectangles. The situation is depicted in figure 9.

### 5.4. Relation with coherent states

Since the orbit space for the $j = 1/2$ representation of $SU(2)$ consists of one single point, this orbit which is the whole of $CP^1$ has got to coincide with the set of coherent states (30) for $j = 1/2$. Indeed one can explicitly work out (30) to get

$$|z\rangle = \frac{1}{\sqrt{1+|z|^2}}\begin{bmatrix}1\\ z\end{bmatrix}. \tag{68}$$

The limit $z \to \infty$ defines one single point in projective space, meaning that the complex plane plus this point indeed forms a two-sphere. Setting $z = \tan\alpha\,\mathrm{e}^{\mathrm{i}\beta}$ one gets a standard parametrization of $CP^1$

$$|\alpha, \beta\rangle = \begin{bmatrix}\cos\alpha\\ \sin\alpha\,\mathrm{e}^{\mathrm{i}\beta}\end{bmatrix} \tag{69}$$

and it is easy to check that each such vector is an eigenvector of $\hat{r}\cdot\vec{J}$ in the direction (55).

The two orbits (53) and (54) of the representation $j = 1$ are the only two-dimensional sets of coherent states of this representation and they must therefore coincide with the sets of coherent states (29) of section 3 for $j = 1$, whose explicit forms are

$$|z; 1\rangle = \frac{1}{1+|z|^2}\begin{bmatrix}1\\ \sqrt{2}z\\ z^2\end{bmatrix} \tag{70}$$

$$|z; 0\rangle = \frac{1}{1+4|z|^2}\begin{bmatrix}\sqrt{2}z^*\\ 1\\ \sqrt{2}z\end{bmatrix} \tag{71}$$

$$|z; -1\rangle = \frac{1}{1+|z|^2}\begin{bmatrix}z^{*2}\\ \sqrt{2}z^*\\ 1\end{bmatrix}. \tag{72}$$

The set (72) coincides with (70) apart from a phase $z^*/z$ after the redefinition $z \to 1/z^*$, which in turn coincides with (53) for $z = \tan\alpha\,\mathrm{e}^{\mathrm{i}\beta}$. The set (71) coincides with (54) for $|z| = \sin\alpha/\sqrt{2\cos(2\alpha)}$ and $\arg z = \beta$.

## 6. Coherent states invariants in particle mechanics

The general result that Hilbert space is uniquely decomposed into orbits of the group generating the coherent states is still valid in particle mechanics. The orbits of the Heisenberg–Weyl group are the sets $\mathcal{C}_\phi$ given in (19). But the non-compactness of the Heisenberg–Weyl group and the infinite-dimensionality of Hilbert space make the method of finding invariants on orbits outlined in section 4 inapplicable. We shall therefore proceed in a different direction.

From (13) and (14) we compute

$$\langle q, p; \phi|Q|q, p; \phi\rangle = \langle\phi|U^+(q, p)QU(q, p)|\phi\rangle = \langle\phi|Q|\phi\rangle + q \tag{73}$$

$$\langle q, p; \phi|P|q, p; \phi\rangle = \langle\phi|U^+(q, p)PU(q, p)|\phi\rangle = \langle\phi|P|\phi\rangle + p. \tag{74}$$

This means that within each set $\mathcal{C}_\phi$ all possible values of $\overline{Q}$ and of $\overline{P}$ are present. Moreover it means that for any two distinct vectors $|\phi'\rangle, |\phi''\rangle \in \mathcal{C}_\phi$ one has $\overline{Q}(\phi') \neq \overline{Q}(\phi'')$ or $\overline{P}(\phi') \neq \overline{P}(\phi'')$. Thus one can use $\overline{Q}$ and $\overline{P}$ as labels for the different vectors in $\mathcal{C}_\phi$. This corresponds to take as fiducial vector $|\phi\rangle$ in $\mathcal{C}_\phi$ the unique vector for which $\overline{Q}(\phi) = \overline{P}(\phi) = 0$. Then

$$\langle p, q; \phi|Q|p, q; \phi\rangle = q \qquad \text{and} \qquad \langle p, q; \phi|P|p, q; \phi\rangle = p \tag{75}$$

as with the Glauber states (15). There the vacuum $|0\rangle$ is the unique vector for which $\overline{Q} = \overline{P} = 0$. Equations (75) also make clear that the little group is trivial (the identity) everywhere in projective space.

We notice that

$$U^+(q, p)(Q - \overline{Q})U(q, p) = Q + q - \overline{Q} = Q \tag{76}$$

$$U^+(q, p)(P - \overline{P})U(q, p) = P + p - \overline{P} = P. \tag{77}$$

Therefore the functions

$$\begin{aligned} M^{mn} &= \langle q, p; \phi| \left\{(Q - \overline{Q})^m, (P - \overline{P})^n\right\} |q, p; \phi\rangle \\ &= \langle\phi|U^+(q, p) \left\{(Q - \overline{Q})^m, (P - \overline{P})^n\right\} U(q, p)|\phi\rangle \\ &= \langle\phi|\left\{\left[U^+(q, p)(Q - \overline{Q})U(q, p)\right]^m, \left[U^+(q, p)(P - \overline{P})U(q, p)\right]^n\right\}|\phi\rangle \\ &= \langle\phi| \left\{Q^m, P^n\right\} |\phi\rangle \end{aligned} \tag{78}$$

with $m$ and $n$ non-negative integers that are invariants within $\mathcal{C}_\phi$. Here $\{,\}$ stands for the anti-commutator. We use it in order to make the functions $M^{mn}$ real since any other ordering of the operators $Q$ and $P$ in (78) can be written in terms of $M^{mn}$ using the canonical commutator $[Q, P] = \mathrm{i}\hbar$. These functions resemble moments of a two-dimensional probability distribution, though their interpretation and properties are different.

The values of $M^{mn}$ do not range independently over the entire real line. Besides the fact that for $m$ and $n$ being even one has $M^{mn} \geqslant 0$, the $M^{mn}$ are still subject to Heisenberg-like inequalities. These look reminiscent of the semi-algebraic variety nature of orbit space in the case of finite-dimensional Hilbert spaces.

The relevant functions in (78) are actually the ones for which the integers $m$ and $n$ satisfy $m + n > 1$ since $M^{00} = 1$ is simply the normalization condition and $M^{01} = M^{10} = 0$ by construction. The 'second-order moments' are the familiar variances and covariance

$$M^{20} = \Delta Q^2 \qquad M^{02} = \Delta P^2 \qquad M^{11} = \sigma_{QP} \tag{79}$$

and the Robertson inequality [36] (a stronger statement then the Heisenberg inequality) reads

$$M^{20}M^{02} \geqslant \tfrac{1}{4}\left[(M^{11})^2 - \hbar^2\right]. \tag{80}$$

For the Glauber states the values of the 'moments' involved in this inequality are easy to compute

$$M^{20} = M^{02} = \hbar/2 \qquad M^{11} = 0 \tag{81}$$

confirming that they are minimum-uncertainty states. It is often not stressed that these states not only have a minimum value for the uncertainty as they also have constant and identical values for the products involved in the uncertainty relation, the standard deviations of $Q$ and $P$, the same happening for all 'moments' of higher order. For any $M^{mn}$ one can write the operator to be averaged $\{Q^m, P^n\}$ in terms of the creation and annihilation operators $a$ and $a^+$. It is the sum of a finite number of products in $a$ and $a^+$

$$\{Q^m, P^n\} = \sum_{i=0}^{m+n} \sum_{j=\text{perm.}} \alpha_{ij} \mathcal{M}_j[a^i (a^+)^{m+n-i}] \tag{82}$$

where the index $j$ runs over the permutations $\mathcal{M}_j$ of operator ordering in $a$ and $a^+$. We have then for the Glauber states

$$M^{mn} = \sum_{i=0}^{m+n} \sum_{j=\text{perm.}} \alpha_{ij} \langle 0|\mathcal{M}_j[a^i (a^+)^{m+n-i}]|0\rangle \tag{83}$$

which is a finite sum of finite terms and which is consequently convergent for any integer values of $m$ and $n$.

This same argument can be used to demonstrate that all $M^{mn}$ converge for sets of coherent states generated from any eigenstate of the number operator $|\phi\rangle = |n\rangle$. And the same is true for any finite combination of eigenvectors of the number operator

$$|\phi\rangle = \sum_{n=0}^{N} \alpha_n |n\rangle. \tag{84}$$

Incidentally these states seem to correspond to the 'undistorted normalizable wavepackets with classical motion' of the harmonic oscillator [37].

The functions (78) do not converge on all orbits. For example, normalizability of $\psi(x)$ does not imply the convergence of $\int \mathrm{d}x\ x|\psi(x)|^2$. But the subspace of Hilbert space where all the $M^{mn}$ converge is still composed of the union of entire orbits of the Heisenberg–Weyl group, and one may wonder whether the functions $M^{mn}$ separate the orbits. We leave this issue for future work. For the moment we notice that the $M^{mn}$ cannot separate a function $\psi(x)$ with an infinite degenerate zero from another which is identical to it on one side of the zero but which flips sign on the other (I thank Gerard 't Hooft for this remark).

## Acknowledgments

I am grateful to Ingemar Bengtsson for discussions concerning almost all parts of this work. Supported by grant no. PRODEP-Acção 5.2.

## Appendix A. Orbits in real representations

This appendix is taken from [26, 38] (sometimes literally) where the authors consider real finite-dimensional and orthogonal linear representations of compact groups.

There is a finite number of orbit types. Strata are smooth disjoint submanifolds of $\mathcal{H}$. However they are not usually patched together smoothly so that the orbit space $\mathcal{H}/G$ is not generally a manifold, but rather is a connected semi-algebraic subvariety of $\mathcal{H}$, that is a subset of $\mathcal{H}$ defined by polynomial equalities and inequalities. The origin $|\psi\rangle = 0$ is an unique orbit with little group $G$, and it belongs to the maximal orbit type.

For compact groups it can be shown that most of the orbits lie on a unique stratum of minimum-orbit type called the principal stratum.

### *A.1. Principal orbit theorem*

*The set of principal vectors is open and dense in $\mathcal{H}$; it is also connected if G is connected. The set of principal orbits is open, dense and connected (even if G is disconnected) in $\mathcal{H}/G$. All principal orbits (vectors) lie in a unique stratum whose orbit type is minimal in the set of orbit types.*

From this theorem it can be shown that the boundaries of the principal stratum either in orbit space $\mathcal{H}/G$ or in $\mathcal{H}$ are disjoint unions of the remaining strata which turn out to be lower-dimensional manifolds. The dimension of the little group is the same all over the principal stratum, $\dim G_p$, and the dimension of orbit space is given by

$$\dim(\mathcal{H}/G) = \dim \mathcal{H} - \dim G + \dim G_p \tag{A.1}$$

where $G_p$ is the little group of the principal vectors.

If $G$ is compact it can be shown that $G$-invariant functions separate the orbits, that is for two distinct orbits there is at least one $G$-invariant function taking different values on them. The set $P^G_{\mathcal{H}}$ of all the real polynomials in $|\psi\rangle$ (that is in its $n$ coordinates, $n$ being the dimension of the vector space $\mathcal{H}$) is a ring under addition and multiplication. An integrity basis $P_i(|\psi\rangle)$ is a discrete subset of $P^G_{\mathcal{H}}$ which generates the ring $P^G_{\mathcal{H}}$ in the sense that any element $P \in P^G_{\mathcal{H}}$ can be written as

$$P(|\psi\rangle) = P[P_i(|\psi\rangle)]. \tag{A.2}$$

The ring of polynomial invariants $P^G_{\mathcal{H}}$ is finitely generated according to the following theorem.

### *A.2. Hilbert's theorem*

*Let G be a compact Lie group acting orthogonally on $\mathcal{H}$. Then $P^G_{\mathcal{H}}$ admits a finite-integrity basis.*

An integrity basis can always be chosen to be minimal, in the sense that no proper subset of it is still an integrity basis. When the polynomials in the minimal-integrity basis are algebraically independent the basis is said to be free and the representation $U(g)$ is said to be co-free.

It can be shown that minimal-integrity basis separate the orbits. This ensures that the set of its elements can be used to parametrize the points in orbit space. $N$ being the number of elements of the integrity basis one can think of the orbits as points in $R^N$ whose coordinates are the elements of the basis. The image of orbit space is typically not the whole $R^N$. For co-free representations $N = \dim \mathcal{H}/G$ and the image of orbit space is a subset of $R^N$ defined through inequalities between the coordinates as happens with a polyhedron.

Let $\{P_i\}$ with $i = 1, \ldots, N$ be a minimal-integrity basis and define the symmetric matrix

$$\hat{P}_{ij} = \vec{\nabla} P_i \cdot \vec{\nabla} P_j \tag{A.3}$$

where the inner product is performed with the same metric used for the inner product $\langle\psi|\psi'\rangle$. Since this inner product is $G$-invariant, the elements of $\hat{P}_{ij}$ are $G$-invariant functions and according to Hilbert's theorem polynomials in the $\{P_i\}$. The following important result holds.

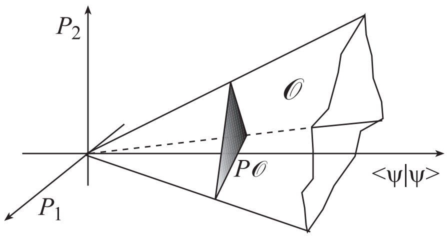


**Figure A.1.** The orbit space for a representation on a vector space and the orbit space for the projective representation associated with the vector space.

*A.3. Theorem*

*The image of orbit space is the subset $\mathcal{O}$ of $R^N$ where $\hat{P}_{ij}$ is positive semi-definite (all its eigenvalues are non-negative). The subset of $\mathcal{O}$ where $\hat{P}_{ij}$ has rank $k$ is the union of all the $k$-dimensional strata, each of them being a connected component of the subset. In particular the subset of $\mathcal{O}$ where the rank of $\hat{P}_{ij}$ is maximal, that is equal to* $\dim \mathcal{H}/G$, *is the image of the principal stratum and is connected.*

We finish with some remarks concerning projective representations, that is the case when one considers the representation space not to be the whole space $\mathcal{H}$ but the projective space $P\mathcal{H}$ of rays in $\mathcal{H}$ (see appendix B; here we consider $\mathcal{H}$ to be real). Since $U(g)$ is linear, $G_\phi$ depends only on the direction of $|\phi\rangle$

$$G_{\alpha|\phi\rangle} = G_{|\phi\rangle} \qquad \text{for} \quad \alpha \neq 0. \tag{A.4}$$

This means that any two vectors lying on the same ray have the same orbit type. Therefore the orbits in $\mathcal{H}$ are infinite copies along each ray of the orbits in projective space $P\mathcal{H}$ plus the origin $|\psi\rangle = 0$. For groups with no fixed points (apart from the origin $|\psi\rangle = 0$) the $G$-invariant $\langle\psi|\psi\rangle \in R_+$ can always be taken to be one of the elements of the minimal-integrity basis. Then one can write

$$\mathcal{O} = P\mathcal{O} \times R_+ + \{|0\rangle\} \tag{A.5}$$

where $P\mathcal{O}$ stands for the image of the orbit space of the projective representation. It turns out that most of the results of this section go through unchanged, particularly in what concerns the geometry of orbit space. The situation is depicted in figure A.1. Of course the use of minimal-integrity basis needs to be adapted. A detailed study of orbit spaces for projective representations can be found in [38]. For our purposes it suffices to mention that whenever necessary, such as in the application of the last theorem of this section one can always start with the vector space representation and fix $\langle\psi|\psi\rangle = 1$ *a posteriori*.

## Appendix B. Complex projective space

Two vectors in Hilbert space $\mathcal{H}$ differing by a multiplicative non-zero complex constant $\alpha$ represent the same physical state

$$|z'\rangle \sim |z\rangle \qquad \text{if} \quad |z'\rangle = \alpha|z\rangle. \tag{B.1}$$

Therefore the space of physical states is the space of rays in Hilbert space or projective space $P\mathcal{H}$, that is the space of equivalence classes defined by (B.1) excluding the vector $|\psi\rangle = 0$. The projective spaces constructed from finite-dimensional Hilbert spaces are called $CP^N$ and are well studied spaces [39]. The superscript $N$ stands for their complex dimension which is one unit lower than the complex dimension of the Hilbert space from which they are constructed.

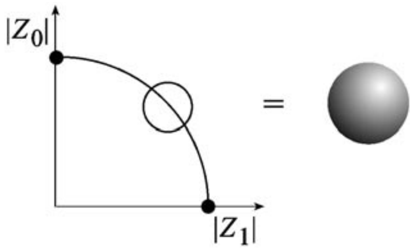


**Figure B.1.** Complex projective space $CP^1$.

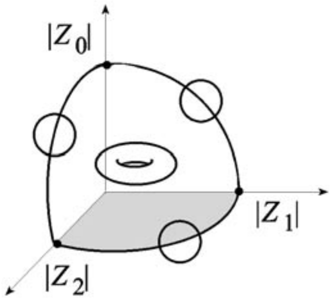


**Figure B.2.** Complex projective space $CP^2$. The shaded region is the vertical projection of the octant used in figure 7.

If $|n\rangle$ is a basis for $(N+1)$-dimensional Hilbert space any vector $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{n=0}^{N} Z_n |n\rangle. \tag{B.2}$$

The complex numbers $Z_n$ are homogeneous coordinates in $\mathcal{H}$ and they can also be used as coordinates in $CP^N$ provided one makes the identifications

$$Z'_n \sim Z_n \qquad \text{if} \quad \exists\alpha : \forall n \qquad Z'_n = \alpha Z_n. \tag{B.3}$$

To form a picture of how $CP^N$ looks like topologically one may consider the $(N+1)$-dimensional space spanned by the absolute values of the homogeneous coordinates $Z_i$ and set $\sum_{i=0}^{N} |Z_i|^2 = 1$. The resulting hyper-surface is the arch that bounds a quadrant for $N = 1$, the curved surface of an octant for $N = 3$, etc. These hyper-surfaces have a natural decomposition in smooth sets of all dimensions from $N$ down to 0. For example, in the case of the octant they are: the face, the three edges and the three vertices. At each point on the interior of the hyper-surfaces (that we may call $N$-octants) sits an $N$-torus because $|Z_n| \neq 0, \forall n$ and the number of relative phases is the maximum $N$. And on each one of the smooth sets mentioned before of dimension $d$ sits a $d$-dimensional torus because $N - d$ of the $|Z_n|$ vanish. In particular the vertices in this picture are points in $CP^N$ and not projections of tori. The lowest-dimensional $CP^0$ is obviously nothing but a point. The situation is depicted in figures B.1 and B.2 for $N = 1$ and 2 respectively. We note that these pictures of $CP^N$ are more than merely topological. For example, geodesics on $CP^N$ with respect to the Fubini–Study metric [39], when projected to the $N$-octant, coincide with the ordinary geodesics on the $N$-sphere, that is, they are the archs of the greater circles (equators) contained in the $N$-octant.

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